Suggested Solutions to:
Resit Exam, Fall 2017
Contract Theory
February 20, 2018

This version: March 2, 2018

Question 1: Adverse selection model with endogenous types

(a) How does $\underline{q}^{SB}$ relate to $\overline{q}^{FB}$? How does $\underline{q}^{SB}$ relate to $\underline{q}^{FB}$? Solve as much as you need of the problem in order to answer those questions. You do not need to show that the second-order condition is satisfied (and you will not get any credit if you nevertheless do that), but otherwise you should prove all your claims.

The principal's problem is to choose $\left(\underline{q}, \overline{q}, \underline{t}, \overline{t}\right)$ so as to maximize

$$V\left(\underline{t}, \underline{q}, \overline{t}, \overline{q}\right) = \nu_1 \left[S\left(\underline{q}\right) - \underline{t}\right] + (1 - \nu_1)\left[S\left(\overline{q}\right) - \overline{t}\right],$$

subject to

$$\overline{t} - C\left(\overline{q}, \overline{\theta}\right) \geq 0, \quad \text{(IR-bad)}$$

$$\underline{t} - C\left(\underline{q}, \underline{\theta}\right) \geq 0, \quad \text{(IR-good)}$$

$$\overline{t} - C\left(\overline{q}, \overline{\theta}\right) \geq \underline{t} - C\left(\underline{q}, \overline{\theta}\right), \quad \text{(IC-bad)}$$

$$\underline{t} - C\left(\underline{q}, \underline{\theta}\right) \geq \overline{t} - C\left(\overline{q}, \underline{\theta}\right), \quad \text{(IC-good)}$$

$$\nu_1 \left[\underline{t} - C\left(\underline{q}, \underline{\theta}\right)\right] + (1 - \nu_1)\left[\overline{t} - C\left(\overline{q}, \overline{\theta}\right)\right] - \psi \geq \nu_0 \left[\underline{t} - C\left(\underline{q}, \underline{\theta}\right)\right] + (1 - \nu_0)\left[\overline{t} - C\left(\overline{q}, \overline{\theta}\right)\right], \quad \text{(IC-effort)}$$

$$\nu_1 \left[\underline{t} - C\left(\underline{q}, \underline{\theta}\right)\right] + (1 - \nu_1)\left[\overline{t} - C\left(\overline{q}, \overline{\theta}\right)\right] - \psi \geq 0. \quad \text{(IR-ante)}$$

In order to solve this problem, we can first note that the constraints IR-good and IR-ante are implied by other constraints in the problem. In particular, we have:

Claim 1: *IC-good and IR-bad imply IR-good.*

Proof. We can write

$$\underline{t} - C\left(\underline{q}, \underline{\theta}\right) \geq \overline{t} - C\left(\overline{q}, \underline{\theta}\right) \geq \overline{t} - C\left(\overline{q}, \overline{\theta}\right) \geq 0,$$

where the first inequality follows from IC-good and the second inequality follows from $\overline{\theta} > \underline{\theta}$ and $C_\theta\left(\overline{q}, \theta\right) > 0$. It follows that $\underline{t} - C\left(\underline{q}, \underline{\theta}\right) \geq 0$. □

Claim 2: *IC-effort, IR-bad, and IR-good imply IR-ante.*

Proof. We can write

$$\nu_1 \left[\underline{t} - C\left(\underline{q},\underline{\theta}\right)\right] + (1-\nu_1)\left[\overline{t} - C\left(\overline{q},\overline{\theta}\right)\right] - \psi \geq \nu_0 \left[\underline{t} - C\left(\underline{q},\underline{\theta}\right)\right] + (1-\nu_0)\left[\overline{t} - C\left(\overline{q},\overline{\theta}\right)\right] \geq 0,$$

where the first inequality follows from IC-effort and the second inequality follows from IR-bad and IR-good. It follows that IR-ante must hold. □

We now have four constraints left. The Lagrangian associated with the maximization problem is:

$$\begin{aligned}
\mathcal{L} &= \nu_1 \left[S\left(\underline{q}\right) - \underline{t}\right] + (1-\nu_1)\left[S\left(\overline{q}\right) - \overline{t}\right] + \lambda\left[\overline{t} - C\left(\overline{q},\overline{\theta}\right)\right] \\
&\quad + \overline{\mu}\left[\overline{t} - \underline{t} - C\left(\overline{q},\overline{\theta}\right) + C\left(\underline{q},\overline{\theta}\right)\right] + \underline{\mu}\left[\underline{t} - \overline{t} - C\left(\underline{q},\underline{\theta}\right) + C\left(\overline{q},\underline{\theta}\right)\right] \\
&\quad + \xi\left[(\nu_1-\nu_0)\left[\underline{t} - \overline{t} - C\left(\underline{q},\underline{\theta}\right) + C\left(\overline{q},\overline{\theta}\right)\right] - \psi\right],
\end{aligned}$$

where $\lambda \geq 0$, $\overline{\mu} \geq 0$, $\underline{\mu} \geq 0$, and $\xi \geq 0$ are the shadow prices associated IR-bad, IC-bad, IC-good, and IC-effort, respectively. Now compute the first-order conditions with respect to each one of the four choice variables:

$$\frac{\partial \mathcal{L}}{\partial \underline{t}} = -\nu_1 - \overline{\mu} + \underline{\mu} + \xi(\nu_1 - \nu_0) = 0, \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial \overline{t}} = -(1-\nu_1) + \lambda + \overline{\mu} - \underline{\mu} - \xi(\nu_1 - \nu_0) = 0, \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \underline{q}} = \nu_1 S'\left(\underline{q}\right) + \overline{\mu} C_q\left(\underline{q},\overline{\theta}\right) - \underline{\mu} C_q\left(\underline{q},\underline{\theta}\right) - \xi(\nu_1-\nu_0) C_q\left(\underline{q},\underline{\theta}\right) = 0, \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial \overline{q}} = (1-\nu_1) S'\left(\overline{q}\right) - \lambda C_q\left(\overline{q},\overline{\theta}\right) - \overline{\mu} C_q\left(\overline{q},\overline{\theta}\right) + \underline{\mu} C_q\left(\overline{q},\underline{\theta}\right) + \xi(\nu_1-\nu_0) C_q\left(\overline{q},\overline{\theta}\right) = 0. \tag{4}$$

By adding up conditions (1) and (2), we obtain $\lambda = 1$. That means, in particular, that IR-bad binds. Next, using (1), condition (3) can be written as

$$\begin{aligned}
\nu_1 S'\left(\underline{q}\right) &= \left[-\overline{\mu} + \underline{\mu} + \xi(\nu_1-\nu_0)\right] C_q\left(\underline{q},\underline{\theta}\right) - \overline{\mu}\left[C_q\left(\underline{q},\overline{\theta}\right) - C_q\left(\underline{q},\underline{\theta}\right)\right] \\
&= \nu_1 C_q\left(\underline{q},\underline{\theta}\right) - \overline{\mu}\left[C_q\left(\underline{q},\overline{\theta}\right) - C_q\left(\underline{q},\underline{\theta}\right)\right].
\end{aligned} \tag{5}$$

Furthermore, using $\lambda = 1$ and (1), condition (4) can be written as

$$\begin{aligned}
(1-\nu_1) S'\left(\overline{q}\right) &= (1-\nu_1) C_q\left(\overline{q},\overline{\theta}\right) + \left[\nu_1 + \overline{\mu} - \xi(\nu_1-\nu_0)\right] C_q\left(\overline{q},\overline{\theta}\right) - \underline{\mu} C_q\left(\overline{q},\underline{\theta}\right) \\
&= (1-\nu_1) C_q\left(\overline{q},\overline{\theta}\right) + \underline{\mu}\left[C_q\left(\overline{q},\overline{\theta}\right) - C_q\left(\overline{q},\underline{\theta}\right)\right].
\end{aligned} \tag{6}$$

Claim 3: *Under the assumption stated in eq. (1) in the question, both second-best quantities equal the first-best quantities, $\overline{q}^{SB} = \overline{q}^{FB}$ and $\underline{q}^{SB} = \underline{q}^{FB}$.*

Proof. We first *guess* that IC-bad and IC-good are both lax at the optimum, which means that $\overline{\mu} = \underline{\mu} = 0$ (this guess will be verified later). It then follows immediately from (5) and (6) that $\overline{q}^{SB} = \overline{q}^{FB}$ and $\underline{q}^{SB} = \underline{q}^{FB}$. Moreover, by (1) we have $\xi > 0$, which means that the IC-effort constraint binds. The binding IC-effort yields

$$\underline{t}^* - \overline{t}^* - C\left(\underline{q}^{FB},\underline{\theta}\right) + C\left(\overline{q}^{FB},\overline{\theta}\right) = \frac{\psi}{\nu_1-\nu_0} \Leftrightarrow \underline{t}^* - C\left(\underline{q}^{FB},\underline{\theta}\right) = \frac{\psi}{\nu_1-\nu_0}, \tag{7}$$

where the last equality follows from the binding IR-bad (and the asterisks indicate transfer levels at the candidate second-best optimum). To prove the claim, it remains to check that both IC-bad and IC-good are satisfied, given the conditions stated in eq. (1) in the question. Consider first IC-bad. By plugging $\underline{t}^*$ and $\overline{t}^*$ into this constraint and using (7) and the binding IR-bad, we obtain

$$\overline{t}^* - C\left(\overline{q}^{FB}, \overline{\theta}\right) \geq \underline{t}^* - C\left(\underline{q}^{FB}, \overline{\theta}\right) \Leftrightarrow \frac{\psi}{\nu_1 - \nu_0} \leq C\left(\underline{q}^{FB}, \overline{\theta}\right) - C\left(\underline{q}^{FB}, \underline{\theta}\right),$$

which indeed holds thanks to eq. (1) in the question. Next consider IC-good. Plugging $\underline{t}^*$ and $\overline{t}^*$ into this constraint and using (7) and the binding IR-bad yields

$$\underline{t}^* - C\left(\underline{q}^{FB}, \underline{\theta}\right) \geq \overline{t}^* - C\left(\overline{q}^{FB}, \underline{\theta}\right) \Leftrightarrow \frac{\psi}{\nu_1 - \nu_0} \geq \overline{t}^* - C\left(\overline{q}^{FB}, \overline{\theta}\right) + C\left(\overline{q}^{FB}, \overline{\theta}\right) - C\left(\overline{q}^{FB}, \underline{\theta}\right) \Leftrightarrow$$

$$\frac{\psi}{\nu_1 - \nu_0} \geq C\left(\overline{q}^{FB}, \overline{\theta}\right) - C\left(\overline{q}^{FB}, \underline{\theta}\right),$$

which also holds by eq. (1) in the question. □

That is, we can conclude that under the assumption stated in the question, both second-best quantities equal their first-best levels, in spite of the fact that the principal faces uncertainty about the agent's type.

To answer the question (and get full credit on Q1a), it suffices to show that the second equality in Claim 3, namely $\boxed{\underline{q}^{SB} = \underline{q}^{FB}}$, holds—provided that the student also points out that, due to the relationship $\underline{q}^{FB} > \overline{q}^{FB}$, this also implies that $\boxed{\underline{q}^{SB} > \overline{q}^{FB}}$.

(b) Explain the intuition/the economic logic behind the results that you find. If you think it it sheds light on the logic for the problem here, you are encouraged to relate to other results/arguments that we have studied and discussed in the course.

In the standard adverse selection model, an agent wants to be perceived as a bad type of agent, because such an agent requires a larger payment in order to be compensated for the production costs. For this reason, IC-good binds in the standard model. In the present model we also have an IC-effort constraint. For this to be satisfied, the principal must make the payment to the good type large relative to that of the bad type (to incentivize the costly effort). This makes it more attractive be be perceived as a good type relative to a bad type, an effect that works in the opposite direction to the one described above for the standard model and which made IC-good bind there. If this "opposite-direction effect" that is created by the IC-effort constraint is somewhat strong but not very strong, then it seems plausible that the two effects will cancel each other out and neither one of them will bind at the optimum; if so, we should expect the second-best solution to coincide with the first-best solution, as it is the IC-good constraint that creates the distortion in the standard model. This is indeed what happens for the intermediate levels of the effort cost that are consistent with the assumption in the question.

In the course we studied a phenomenon called countervailing incentives. This arose in an environment in which an agent of a relatively able type also had a relatively attractive outside option. This, too, created an "opposite-direction effect" that could, if the effect was sufficiently strong but not too strong, lead to a second-best solution without any distortion.

Question 2: Moral hazard with mean-variance preferences

(a) Solve for the β parameter in the second-best optimal contract, denoted by β^{SB} (you do not need to solve for α^{SB}, and you will not get any credit if you nevertheless do that). You should make use of the following (well-known) result:

$$EU = -\exp\left[-r\left(\alpha + \beta e - \frac{1}{2}e^2 - \frac{1}{2}\nu r \beta^2\right)\right].$$

- P chooses the parameters in the contract, α and β. In addition, P can effectively choose A's effort e, because P designs the incentives that A faces when deciding what effort to make. We can thus think of P as choosing α, β, and e in order to maximize his expected payoff, subject to A's individual rationality (IR) constraint and incentive compatibility (IC) constraint. P's problem:

$$\max_{\alpha,\beta,e}\left\{\overbrace{(1-\beta)\,e-\alpha}^{=EV}\right\}$$

subject to

$$\overbrace{-\int_{-\infty}^{\infty}\exp\left[-r\left(t-c\left(e\right)\right)\right]f\left(z\right)dz}^{=EU} \geq -\exp\left[-r\widehat{t}\right], \tag{IR}$$

$$e \in \arg\max_{e'} EU\left(e'\right). \tag{IC}$$

The IC constraint says that e indeed maximizes A's utility among all the e's that A could choose. The IR constraint says that A's expected utility if accepting the contract is at least as large as his utility from his outside option; this therefore ensures that A wants to participate.

- The IC constraint above is actually a whole set of infinitely many constraints. In order to reduce these to one single IC constraint, we can make use of the first-order approach, which means that we replace IC above with the first-order condition from A's maximization problem (for some arbitrary values of the contract parameters α and β). From the question we have that A's expected utility can be written as

$$EU = -\exp\left[-r\left(\alpha + \beta e - \frac{1}{2}e^2 - \frac{1}{2}\nu r \beta^2\right)\right].$$

Maximizing EU is equivalent to maximizing a monotone transformation of this expression, so we can without loss of generality let A maximize

$$\widetilde{EU} = \alpha + \beta e - \frac{1}{2}e^2 - \frac{1}{2}\nu r\beta^2. \tag{8}$$

- We have

$$\frac{\partial \widetilde{EU}}{\partial e} = \beta - e = 0$$

Therefore A's optimal effort level is

$$e = \beta. \tag{9}$$

- We can write the IR constraint as

$$
\begin{aligned}
-\int_{-\infty}^{\infty} \exp\left[-r\left(t-c\left(e\right)\right)\right] f\left(z\right) dz \geq -\exp\left[-r\widehat{t}\right] &\Leftrightarrow \\
-\exp\left[-r\left(\alpha+\beta e-\frac{1}{2}e^2-\frac{1}{2}\nu r\beta^2\right)\right] \geq -\exp\left[-r\widehat{t}\right] &\Leftrightarrow \\
\exp\left[-r\left(\alpha+\beta e-\frac{1}{2}e^2-\frac{1}{2}\nu r\beta^2\right)\right] \leq \exp\left[-r\widehat{t}\right] &\Leftrightarrow \\
-r\left(\alpha+\beta e-\frac{1}{2}e^2-\frac{1}{2}\nu r\beta^2\right) \leq -r\widehat{t} &\Leftrightarrow \\
\alpha+\beta e-\frac{1}{2}e^2-\frac{1}{2}\nu r\beta^2 \geq \widehat{t} &\Leftrightarrow \\
\alpha \geq \widehat{t}-\beta e+\frac{1}{2}e^2+\frac{1}{2}\nu r\beta^2 &.
\end{aligned}
$$

Plugging in (9) in this inequality, we obtain

$$
\begin{aligned}
\alpha &\geq \widehat{t}-\beta^2+\frac{1}{2}\beta^2+\frac{1}{2}\nu r\beta^2 \\
&= \widehat{t}-\frac{1}{2}\left(1-\nu r\right)\beta^2.
\end{aligned}
$$

Plugging in (9) into P's objective function $EV = (1-\beta)\,e - \alpha$, we have

$$EV = (1-\beta)\,\beta - \alpha.$$

- Using the above results, P's problem becomes

$$\max_{\alpha,\beta}\left\{(1-\beta)\,\beta-\alpha\right\} \qquad \text{subject to}$$

$$\alpha \geq \widehat{t}-\frac{1}{2}\left(1-\nu r\right)\beta^2. \tag{IR}$$

- It is clear that IR must bind, as the objective is decreasing in α and the constraint is tightened as α is lowered (thus P wants to lower α until the constraint says stop). We thus have $\alpha = \widehat{t}-\frac{1}{2}\left(1-\nu r\right)\beta^2$. Plugging this value of α into the objective yields the following unconstrained problem:

$$\max_{\beta}\left\{\beta-\frac{1}{2}\left(1+\nu r\right)\beta^2-\widehat{t}\right\},$$

with the first-order condition

$$1-\left(1+\nu r\right)\beta = 0 \Rightarrow \beta^{SB} = \frac{1}{1+\nu r}.$$

(b) Does the agent get any rents at the second-best optimum? Do not only answer yes or no, but also explain how you can tell.

- No, he does not get any rents at the second-best optimum. "Rents" are defined as any payoff from accepting the contract that exceeds the outside option payoff. However, we saw under a) that the IR constraint binds at the optimum, which means that A does not get any rents.

(c) The first-best values of the effort level and the β parameter equal $e^{FB} = 1$ and $\beta^{FB} = 0$, respectively. How do these values relate to the corresponding second-best values? In particular, is there under- or overprovision of effort at the second-best optimum?

- We have from the above analysis that $\beta^{SB} = e^{SB} = \frac{1}{1+vr}$. We see that there is underprovision of effort (as $e^{SB} < e^{FB}$). We also see that the β parameter is too large relative to the first best level ($\beta^{SB} > \beta^{FB}$).

(d) Consider the limit case where $r \to 0$. Explain what happens to the relationship between the second-best and the first-best effort levels. Also explain the intuition for this result.

- In the limit where $r \to 0$, A is risk neutral. We see from above that in that limit, $e^{SB} = 1$. That is, the second-best effort level coincides with the first-best level: there is no inefficiency in spite of the fact that there is asymmetric information. The reason why this can occur is that when A is risk neutral he doesn't mind bearing risk. Therefore P can incentivize A very strongly, so that indeed $\beta^{SB} \to 1$ as $r \to 0$: A's compensation depends fully on the stochastic variable, so he makes the same decision as P would have made if he had been in A's job.
- The intuition is the same as we have discussed in other parts of the course, for example in the 2x2 moral hazard model with a risk neutral agent who is not protected by limited liability. There we explained the intuition as follows:
  - The economic meaning of the fact that A is risk neutral is that he cares only about whether his payment t is large enough *on average*. Hence, P can, without violating the participation constraint, incentivize A by giving him a negative payment (in practice a penalty) in case of a low output. More generally, P can achieve the first-best outcome by making A the residual claimant:
    * Then A effectively buys the right to receive any returns: "the firm is sold to the agent".
    * Thereby, the effort level is chosen by the same individual who bears the consequences of the choice.
    * In this situation A makes the same effort choice as P would have made.